Exam IPDS 4 April 2014

-Solutions-

1. (a) $x' = ax - h\sin t$

$x_h' = a x_h \Rightarrow x_h(t) = K e^{at} \quad , K \in \mathbb{R}$

particular solution:

ansatz $x_p(t) = A\cos t + B\sin t$

$\Rightarrow x_p'(t) = -A\sin t + B\cos t$

$\overset{!}{=} aA\cos t + aB\sin t - h\sin t$

equating coefficients of sin and cos gives:

$$-A = aB - h$$
$$B = aA$$

$$\Leftrightarrow \quad -A = a^2 A - h$$
$$B = aA$$

$$\Leftrightarrow \quad A = \frac{h}{1+a^2}$$
$$B = \frac{ah}{1+a^2}$$

$\Rightarrow$ general solution

$$x(t) = x_h(t) + x_p(t)$$
$$= K e^{at} + \frac{h}{1+a^2}\cos t + \frac{ah}{1+a^2}\sin t$$

the unique periodic solution as $K = 0$

(b) $x(0) = x_0$

$$\Rightarrow \quad x_0 = K + \frac{h}{1+a^2}$$

$$\Rightarrow \quad K = x_0 - \frac{h}{1+a^2}$$

$$\Rightarrow \quad x(t) = \left(x_0 - \frac{h}{1+a^2}\right)e^{at} + \frac{h}{1+a^2}(\cos t + a\sin t)$$

$= \phi_t(x_0)$ where ϕ is the flow system is 2π-periodic. So the Poincaré map is given

by $$P(x) = \phi_{2\pi}(x) = \left(x - \frac{h}{1+a^2}\right) e^{a 2\pi} + \frac{h}{1+a^2}$$

fixed point: $P(x) = x$

$$\Leftrightarrow \quad \left(e^{2\pi a} - 1\right) x = \frac{h}{1+a^2} e^{a 2\pi} - \frac{h}{1+a^2}$$

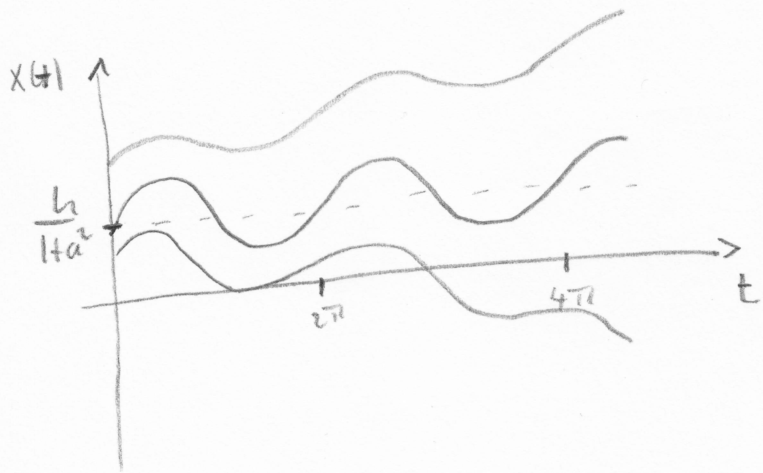
$$\Leftrightarrow \quad x = \frac{h}{1+a^2}$$

stability:

$$P'\left(\frac{h}{1+a^2}\right) = e^{a 2\pi} > 1 \quad \text{since } a > 0$$

$\Rightarrow$ $x = \frac{h}{1+a^2}$ is unique fixed point which is a source

(c)



2. (a) Let $X(t)$ be a non-constant solution.

Then $\frac{d}{dt} V(X(t)) = \nabla V(X(t)) \cdot X'(t) = -\nabla V(X(t)) \cdot \nabla V(X(t))$

$= -|\nabla V(X(t))|^2 < 0$ as $X'(t) = \nabla V(X(t)) \neq 0$

(b) Let X_0 be an equilibrium point.

Then X_0 is called asymptotically stable if X_0 is (Lyapunov) stable, i.e. for each neighbourhood θ of X_0 there exists a neighbourhood θ_1 of X_0 such that for all initial conditions $X(0) \in \theta_1$ it holds that $X(t) \in \theta$ for all $t \geq 0$, and moreover θ_1 can be chosen such that for all $X(0) \in \theta_1$, $X(t) \to X_0$.

(c) Choose V as a Lyapunov function.

If X^* is an isolated minimum then $\nabla V(X^*) = 0$ and there exists a neighbourhood θ such that $\nabla V(X) \neq 0$ for $X \in \theta \setminus \{X^*\}$.

Together with the result from part (a) we see that V is a strict Lyapunov function on θ and hence by the Lyapunov Th^{m} we can infer asymptotic stability of X^*. Moreover θ belongs then to the basin of attraction of X^*.

In particular we can choose θ to be the open region enclosed by a level set $\{X \in \mathbb{R}^n : V(X) < c\}$. (or more precisely the connected component of such a region which contains X^*)

(d) The linearization at X^* is given by

$X' = HX$ where H is the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{pmatrix} \text{ at } X^*.$$

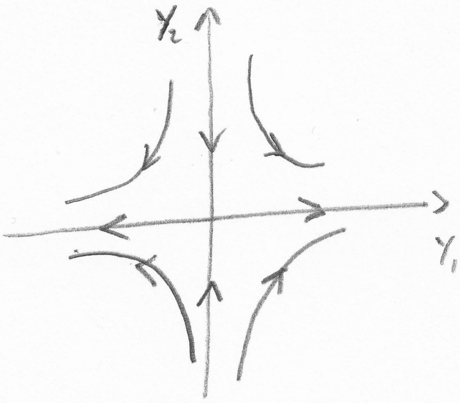
Since H is symmetric its eigenvalues are real. Hence H is hyperbolic if H has only non-vanishing eigenvalues.

Then: if all eigenvalues are negative then X^* is asymptotically stable. If one eigenvalue is positive then X^* is not stable.

3. (a) The matrix defining the linear system is upper triangular. The eigenvalues can hence be read off from the diagonal and are 2 and -1.
The canonical form is

$$y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} y$$

(b) phase portrait canonical form:



phase portrait original system:

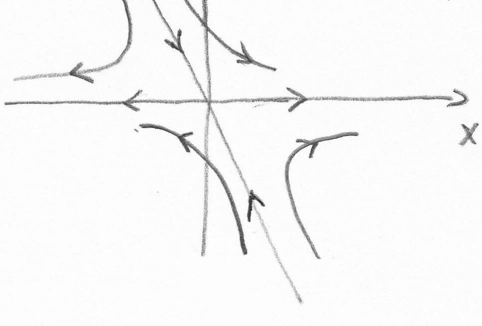
determine eigenvectors:

$$2: \quad \begin{pmatrix} 2-2 & 1 \\ 0 & -1-2 \end{pmatrix} u = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

choose $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$-1: \quad \begin{pmatrix} 2-(-1) & 1 \\ 0 & (-1)-(-1) \end{pmatrix} v = 0 \iff \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

choose $v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(c) We have

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

Let $T = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$

The flow of the original system is

$$\phi_t = \exp\left[\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} t\right]$$

The flow of the canonical system is

$$\phi_t^c = \exp\left[\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} t\right]$$

The flows are conjugate by T:

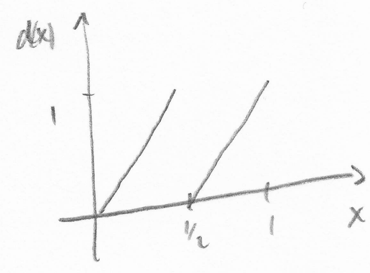
$$\phi_t^c = T^{-1} \phi_t T$$

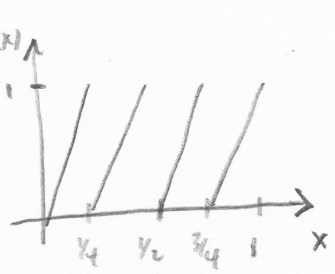
4. (a) Consider $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \dots$

The system is chaotic if

1. periodic orbits are dense, i.e. for all x_0 and each open neighbourhood U of x_0 there exists a periodic point contained in U
2. the system is transitive, i.e. for all open sets U and V there exists $n \in \mathbb{N}$ such that
$$f^n(U) \cap V \neq \emptyset$$
3. the system has sensitive dependence on initial conditions, i.e. $\exists \beta > 0$ such that for all x and open neighbourhood U of x there is $y \in U$ and $n > 0$ such that
$$d(f^n(x), f^n(y)) > \beta$$

(b) consider the graphs of d and its iterates





d^n maps 2^n intervals
$$I_k^n := \left[(k-1)\left(\tfrac{1}{2}\right)^n, k\left(\tfrac{1}{2}\right)^n\right], \quad k = 1, \dots, 2^n$$
surjectively to the interval $[0,1)$

We have $[0,1) = \bigcup_{k=1}^{2^n} I_k^n$ and length of I_k^n equal to $\left(\frac{1}{2}\right)^n$

1. Each interval I_k^n contains a periodic point of period n.

$\Rightarrow$ periodic orbits are dense.

2. Let $U, V \subset [0,1]$ open

$\Rightarrow$ $\exists\, n, k$ such that $I_k^n \subset U$

$\Rightarrow$ $d^n(I_k^n) = [0,1] \cap V \neq \emptyset$

$\Rightarrow$ d is transitive

3. d has sensitive dependence on initial conditions.

Choose $\beta = 2$. $\Rightarrow$

$|d(x) - d(y)| \geq \beta |x - y|$ for all $x, y \in [0, \frac{1}{2}]$ and $x, y \in [\frac{1}{2}, 1]$

The rest is clear.